Knots in Mathematics

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Abstract: Inspired by real world applications, mathematicians have studied knots since the late 18th century. One fundamental question is to find ways to distinguish knots which cannot be transformed into one another without cutting the string. The mathematical notion of *knot invariant* tries to answer this question by assigning a number to a depiction of a knot in a way that if two representations stand for the same knot, then they get assigned the same value. A big breakthrough in this field, with applications to Quantum Physics, was the discovery of the *Jones polynomial* in the 1980s. I will provide concrete examples of knots made from string and try to explain how one computes the abstract Jones polynomial to distinguish them. This should give an idea how mathematicians develop and apply abstract concepts.

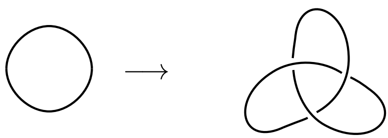
1 What is a Knot? — A Mathematical Definition

Knots have been used in every day life and art since ancient history. Mathematicians started to study knots from the end of the 18th century. Carl Friedrich Gauss was one of the first to develop a systematic theory of knots in mathematics.

There are now different ways to make precise what is meant by a knot using different modern formal mathematical languages such as *Topology* or *Differential Geometry*. Using the later, one can say:

Definition 1. A *knot* is a *smooth* embedding of a circle into three-dimensional euclidean space $\mathbb{R}^3$.

As an example, consider



Is this a good definition? It seems incomplete as we can deform or rotate knots according to this definition and they still represent the same knot. This problem is addressed in the next section.

2 When are two Knots the Same? — an Equivalence Relation

Imagine we have a way of telling whether two knots are essentially the same and say two knots are *equivalent* in this case. This rule should satisfy:

1. Each knot is equivalent to itself (*reflexivity*).

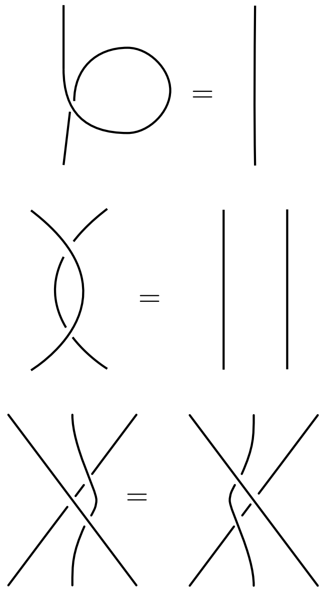
2. If a knot is equivalent to a second one, then the second one is equivalent to the first one (*symmetry*).

3. If a knot is equivalent to another knot, and the second knot is equivalent to a third knot, then the first knot is equivalent to the third one (*transitivity*).

This is another fundamental concept in mathematics, an *equivalence relation*.

Informally, two knots as defined in Section 1 are *equivalent* if the ambient three-dimensional space can be deformed in a continuous matter turning one knot into the other. This is called a *ambient isotopy*. This concept resembles the idea of knotting a piece of string.

An important result by Reidemeister from 1927 [Rei27] states that two knots are only equivalent if they can be transformed into one another using three elementary moves:



3 How to Distinguish Knots? — The Idea of *Invariants*

Imagine a machine where we input knots, and obtain a number on a screen. If this machine gives the same number for equivalent knots then we have an *invariant*. Hence measuring physical properties of knots does not give invariants.

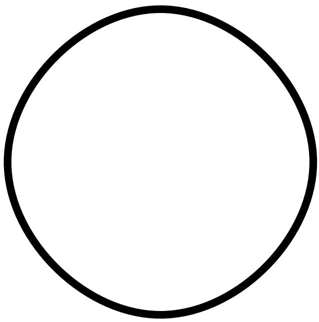
The use of this is, that we can distinguish non-equivalent knots. If two knots give different numbers, then they cannot be equivalent using the logical contrapositive.

The better an invariant, the more knots it distinguishes. Finding these invariants relates to current research in Mathematics.

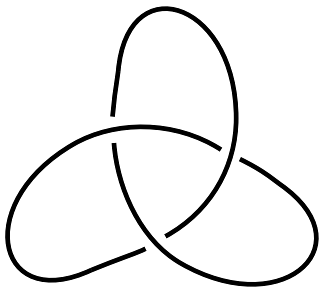
- The *Jones Polynomial* $V(K) \in \mathbb{Z}[t^{1/2}]$ for a knot K was discovered by Vaughan Jones in 1984 [Jon85] and has applications in Quantum Physics due to Edward Witten [Wit89]. It is still an open question if there is a knot which is *not* equivalent to the circle (the *unknot*) but has the same Jones Polynomial, namely 1.

- More refined concepts can distinguish all knots. This relates to *Quantum Groups* and *Braided Monoidal Categories* which are topics relevant for my PhD project. One application of my work can be to obtain new invariants of so-called *ribbons* which are more refined knot where we can distinguish the two sides of a piece of string.

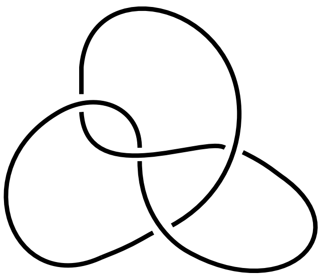
4 Examples of Knots



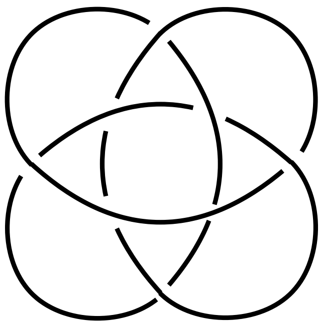
(a) The Unknot K_1



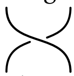
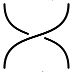
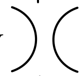
(b) The Trefoil K_2



(c) The Eight K_3



(d) The Bowline K_4

Using the following recursion, one can easily compute the Jones Polynomial $V(L)$ for a knot L. The Jones Polynomial of the unknot K_1 equals 1, i.e. $V(K_1) = 1$. We can take a knot and look at one single crossing. Denote the knot where this is an upper crossing  by L_-, the same knot replacing this one crossing by  is denoted L_+ and the knot where we replace the crossing by  is denoted L_0.

Using this, the following formula enables us to compute the Jones Polynomial for any knot:

$$(t^{1/2} - t^{-1/2})V(L_0) = t^{-1}V(L_+) - tV(L_-).$$

Example 2.

$$\begin{aligned}
V(K_1) &= 1, \\
V(K_2) &= t^{-2} + t^{-3/2} - t^{-1/2}, \\
V(K_3) &= t^{-7/2} + t^{-3} - t^{-5/2} - t^{-2} + t^{-1}, \\
V(K_4) &= t^{-7/2} - t^{-5/2} + t^{-2} + t^{-3/2} + t^{-1} - t^{-1/2} - 1.
\end{aligned}$$

This means that none of the four knots are equivalent. Try to deform one into the other for yourself using the models made from string provided without cutting the string.

It is generally difficult to *prove* that we cannot deform one knot into the other. Invariants provide a powerful tool to answer these questions.

Bibliography

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[Rei27] K. Reidemeister, *Elementare Begründung der Knotentheorie*, Abh. Math. Sem. Univ. Hamburg 5 (1927), no. 1, 24–32. MR3069462

[Wit89] E. Witten, *Quantum field theory and the Jones polynomial*, Braid group, knot theory and statistical mechanics, 1989, pp. 239–329. MR1062429